Introduction to Critical Phenomena

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Overview

- Critical phenomena,
- fractals,
- Conformal Invariance
- Classification of critical phenomena by Conformal Field Theory

General references

Specific references have been given in the main text of my lecture notes but for students who like to read reviews on the topics covered in these lectures the following references are suggested:

Critical Phenomena :

- Sornette, D. *Critical phenomena in natural sciences: chaos, fractals, self-organization and disorder: concepts and tools*. Springer Science & Business Media, 2006.
- Stanley, H. Eugene. "Scaling, universality, and renormalization: Three pillars of modern critical phenomena." *Reviews of modern physics* 71.2 (1999): S358.

General references

Conformal Field Theory

- Schellekens, A. N. "Introduction to conformal field theory." *Fortschritte der Physik/Progress of Physics* 44.8 (1996): 605-705.
- Francesco, Philippe, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Springer Science & Business Media, 2012.

The renormalization group

- Goldenfeld, Nigel. Lectures on phase transitions and the renormalization group. CRC Press, 2018.

Overview

- Critical Phenomena
- Order parameter, spontaneous symmetry breaking
- Ergodicity breaking
- Critical Exponents
- Universality
- Scale invariance
- Renormalization Group

Critical Phenomena

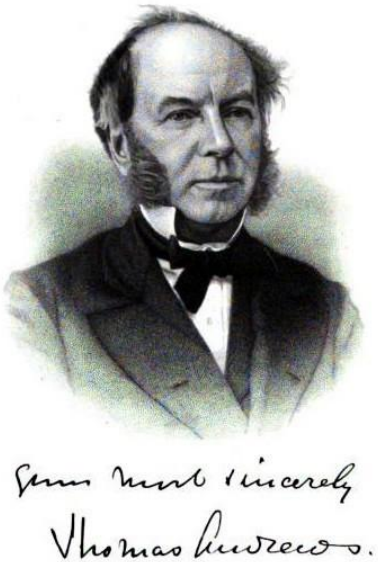
Cagnard de la Tour 1822

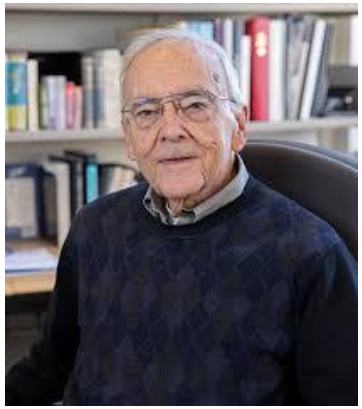
Andrew 1869

Paul Ehrenfest 1900

BenjaminWidom 1960

Wilson 1970



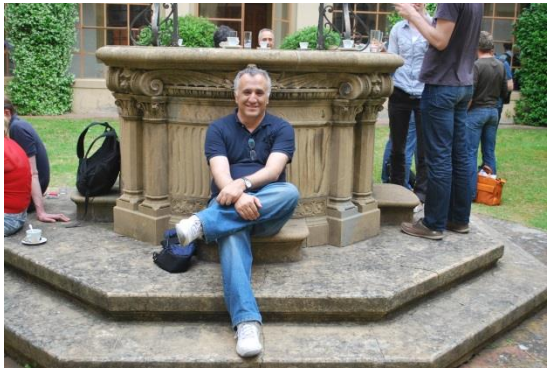


Critical Phenomena

Phase Transitions were classified by Ehrnfest by their “Order” which is the order at which the derivative of free energy becomes singular. Today this classification is not adopted anymore.

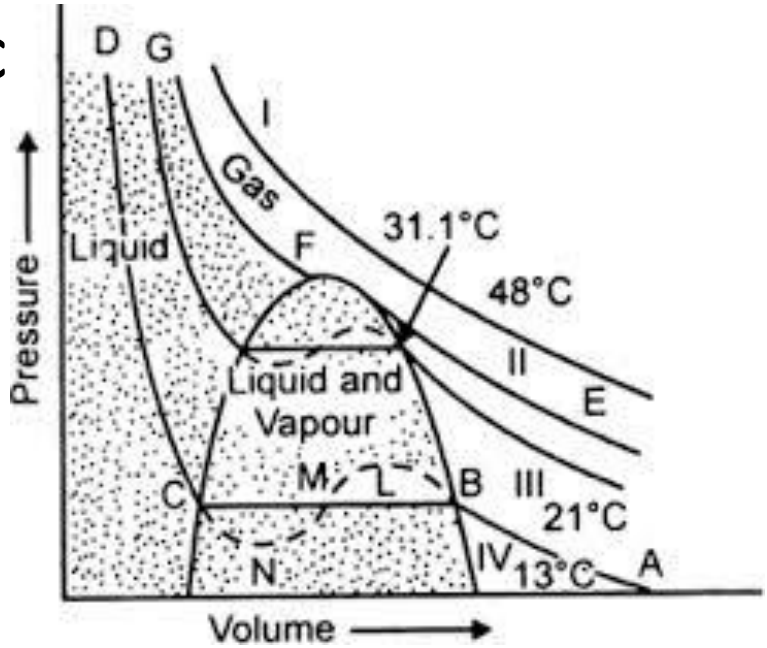
Paul Ehrenfest

1880-1933



Liquid vapor Transition

Matter changes phase
under change of macroscopic
variables such as
temperature,
pressure , magnetization,..

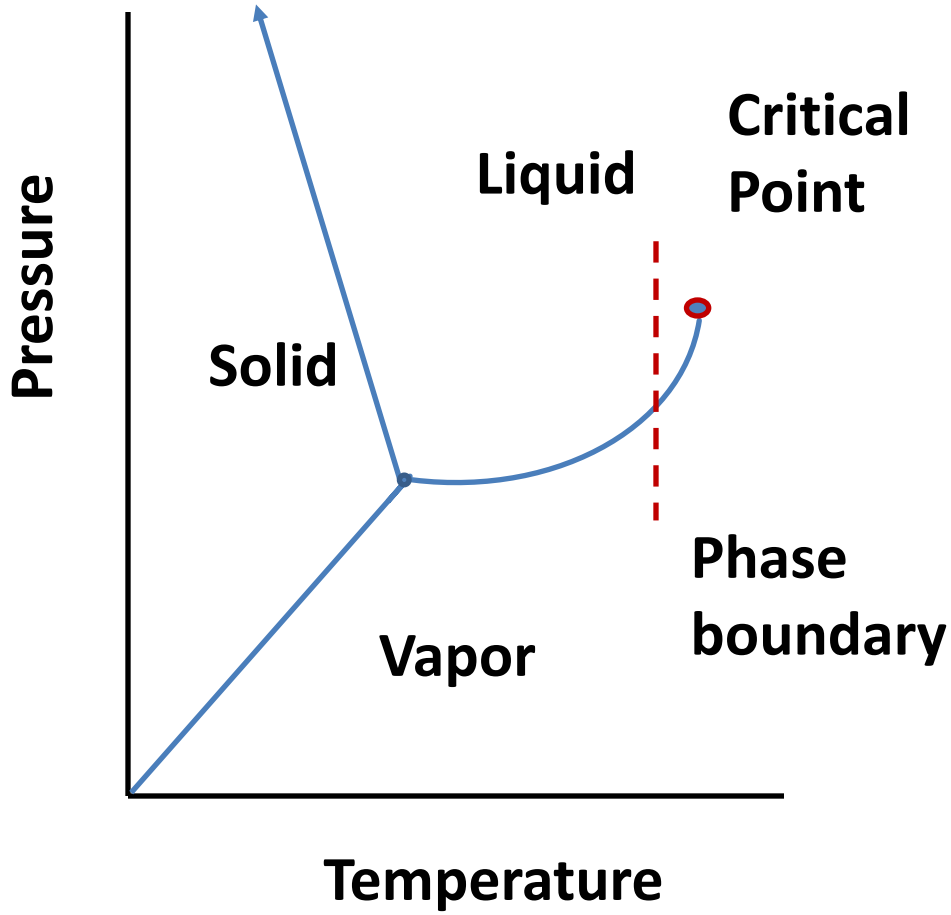


Phase Diagram

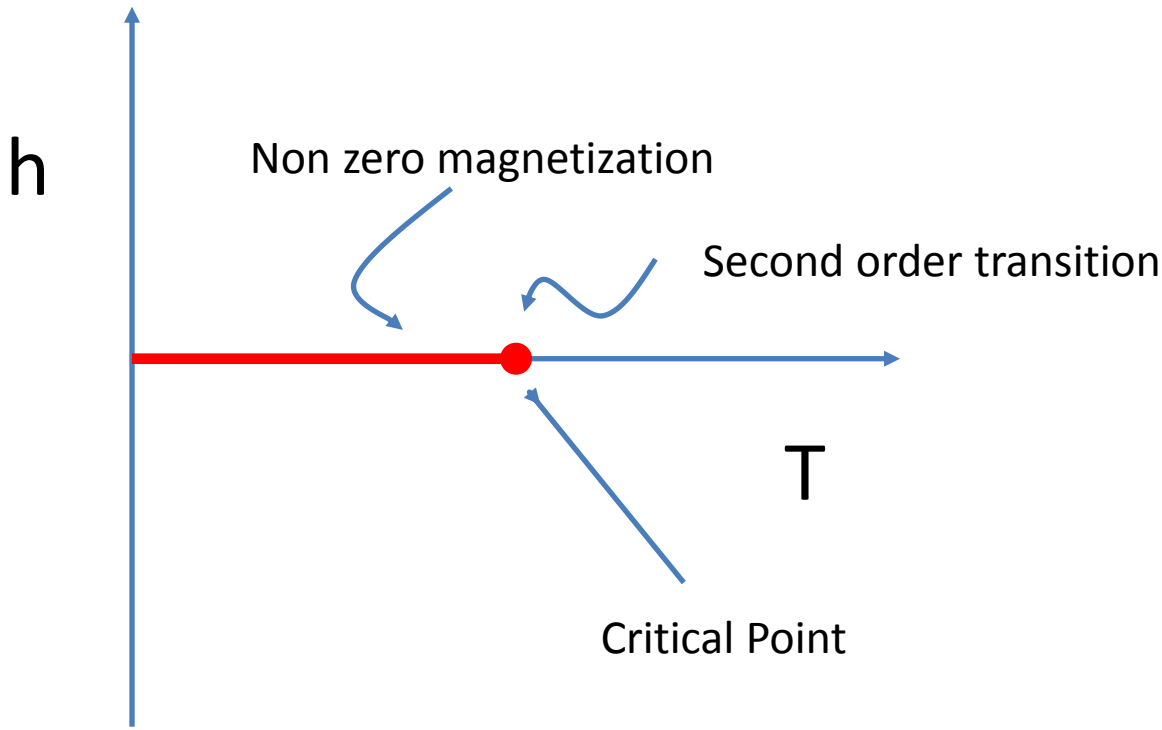
- Matter can exist in many phases

liquid, solid, gas, Paramagnetic , BEC, ..

- These phases come to an end on an interface, this happens due to immiscibility of matter in different phases



Magnetic Phase Diagram



In contrast to magnetization,
the Vapor-Water transition is first order,
the order parameter is the density.
It jumps in value across the phase transition point

Order/Disorder transition

Low Temp	High Temp
More Order	Less Order
Less Symmetry	More Symmetry

Scale invariance at the symmetry breaking point

Even larger symmetry Conformal

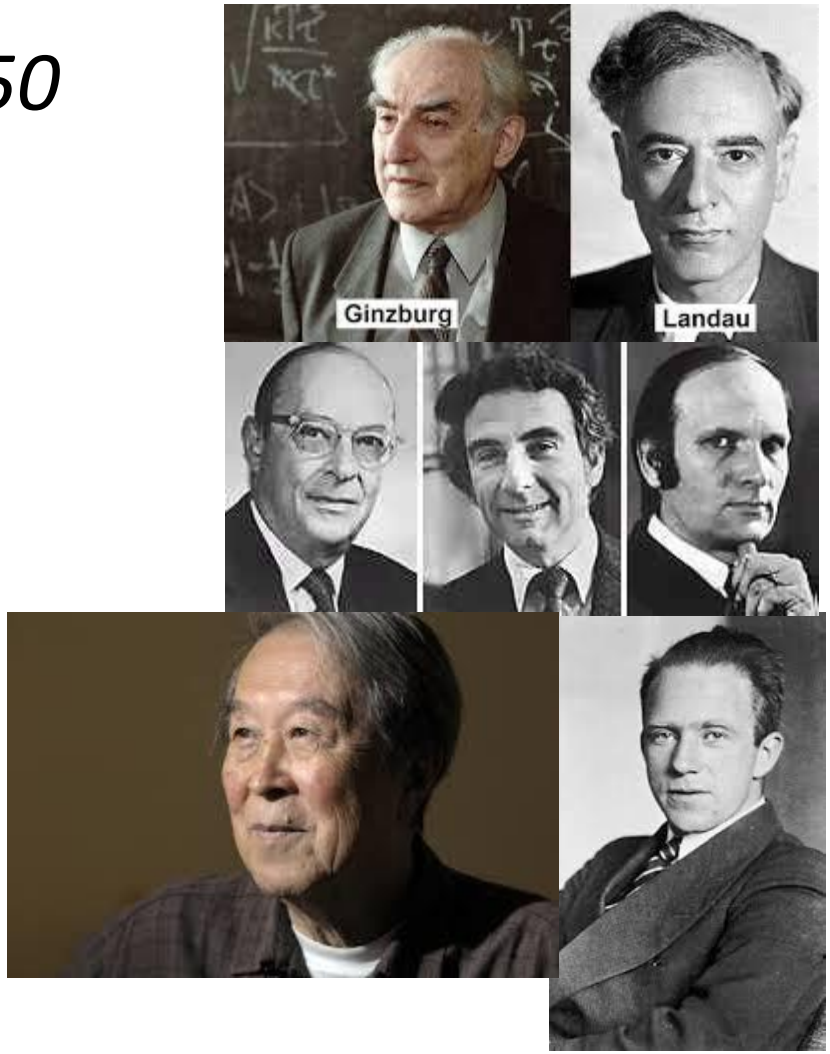
$T=T_c$

Union of three seemingly different concepts

1. Spontaneous Symmetry breaking
2. Ergodicity breaking
3. Renormalization group flow

Spontaneous Symmetry Breaking

- *Ginzburg, Landau 1950*
- *Bardeen, Cooper and Schrieffer 1957*
- *Heisenberg, 1959*
- *Nambu, (1960)*



Spontaneous symmetry breaking (SSB)

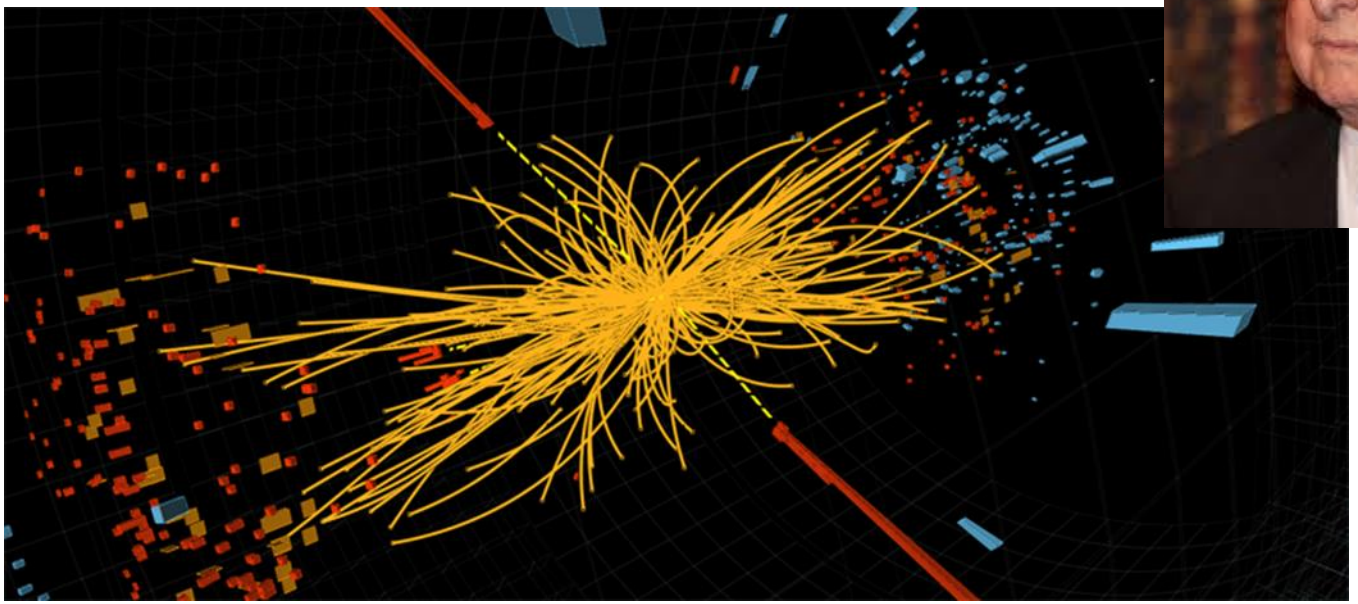
Symmetry operator S: QUANTUM PHYSICS

$$[S,H]=0$$

Vacuum is not symmetric :

Symmetric	Symmetry Broken
$S\|0\rangle = 0$	$S\|0\rangle = \|0'\rangle$

SSB



A proton-proton collision event in the CMS experiment producing two high-energy photons (red towers). An even showing the decay of a Higgs boson but also consistent with background Standard Model physics processes. (simulation, CERN)

SSB

- example of a double well potential is:

$$V(\varphi) = \frac{g}{4}\varphi^4 + \frac{t}{2}\varphi^2$$

For t>0 this potential has one minimum at

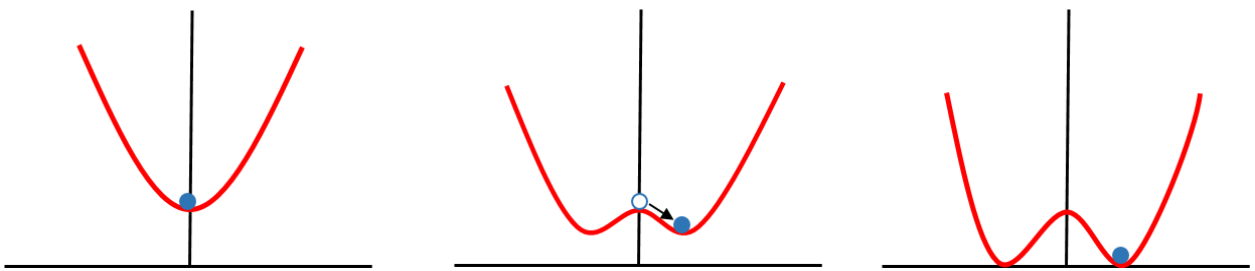
$$\varphi = 0$$

For t<0 has minima at

$$\varphi^2 = \frac{-t}{g}$$

We can choose $t = \frac{T - T_c}{T_c}$, so that the above conditions refer directly to a critical point.

Spontaneous symmetry breaking



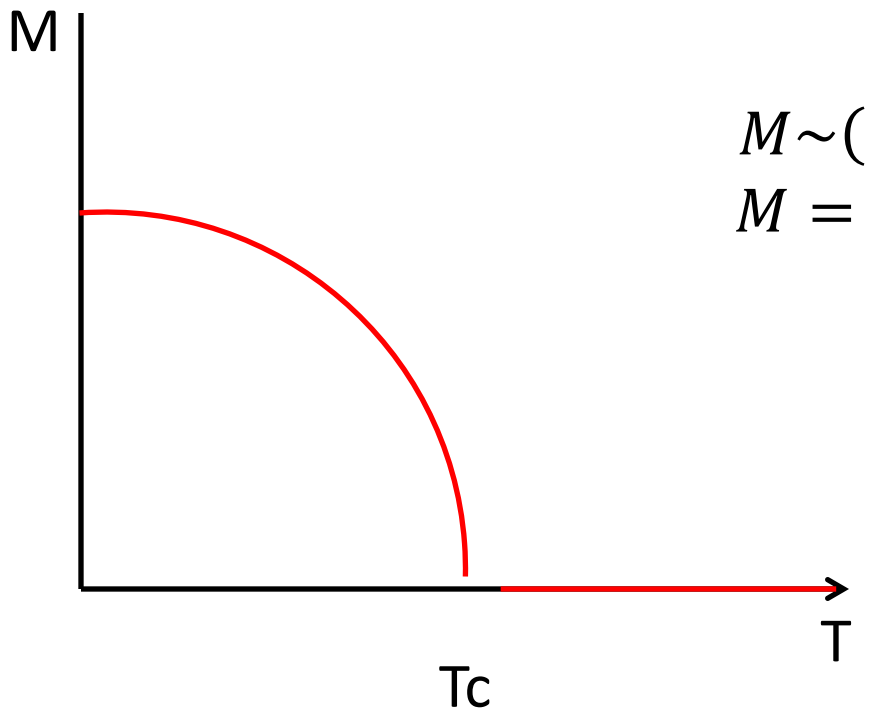
$T > T_c$ $\quad$ $T \lesssim T_c$ $\quad$ $T < T_c$

Order Parameter for 2d Ising model Magnetization

Magnetization is a good order parameter for the Ferromagnetic/ Paramagnetic Transition

For any given critical behavior finding a good order parameter is challenging.

$$M = \langle \varphi \rangle = \frac{1}{N} \sum_n \sigma_i$$



$$M \sim (-t)^{\beta} \ , T < T_c$$
$$M = 0 \qquad , T > T_c$$

Ergodic Theorem

Ergodic Theorem:

Over time the phase space of a dynamical system is uniformly covered.

$$\lim_{t\to\infty} \frac{1}{T} \int_0^T f(T_t x) dt = \frac{1}{\Omega} \iint d\mu \, f(p, q)$$

Therefore ensemble averages can be replaced with time averages.

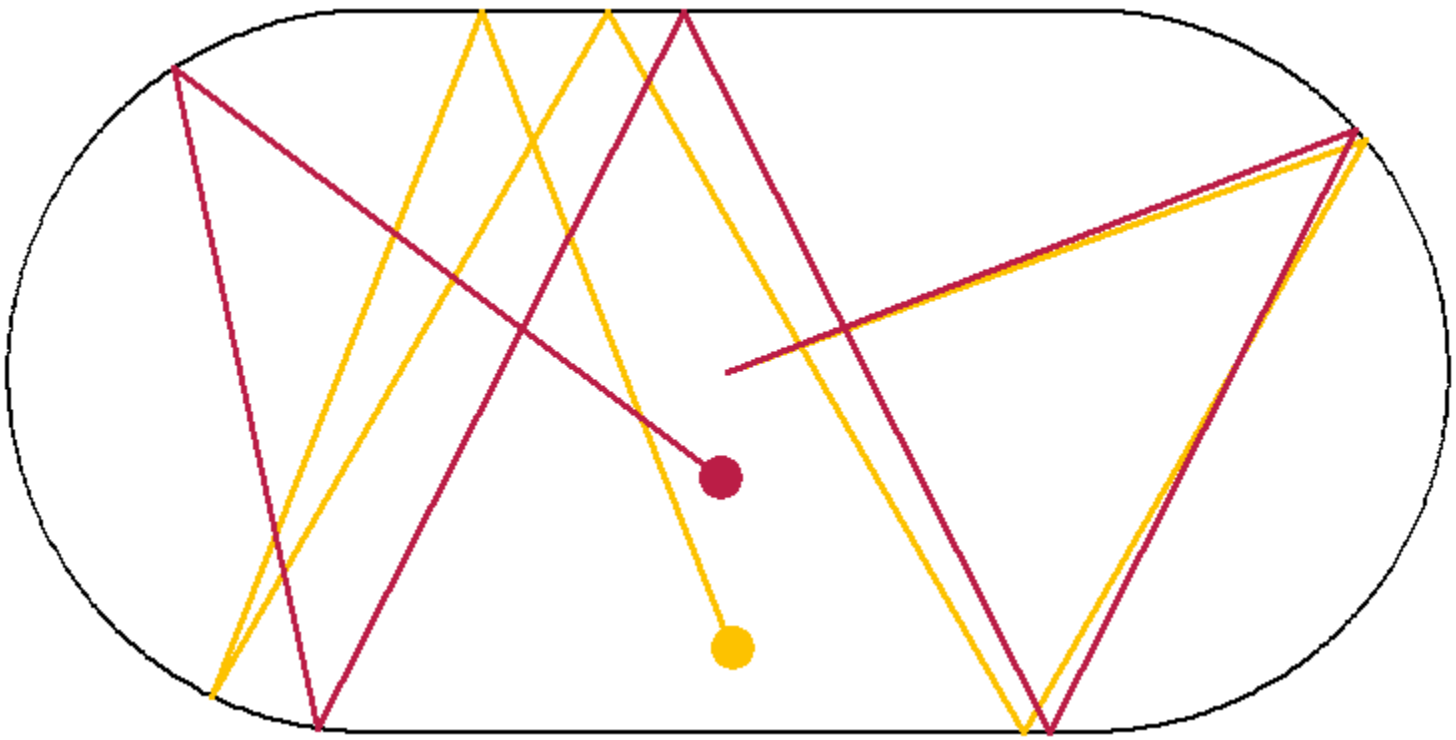
Bunimovich Stadium

$$T_t : \Omega \to \Omega \quad t \epsilon \mathbb{R}$$

with randomly chosen initial position and velocity, over time its position becomes uniformly distributed over the whole stadium.

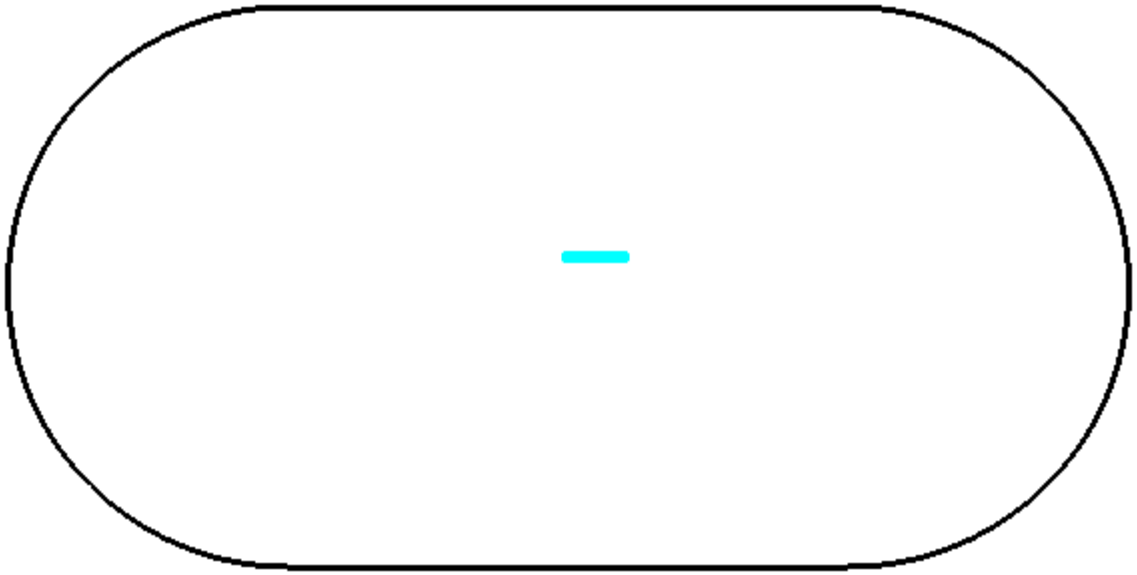
Bunimovich Stadium

Ergodic hypothesis



Leonid A. Bunimovich, On the ergodic properties of nowhere dispersing billiards, Comm. Math. Phys. 65 (1979), 295–312

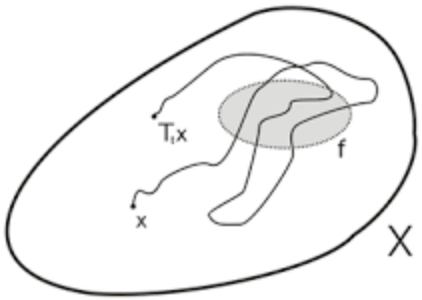
Animation by Phillipe Roux : blogs.ams.org/visualinsight/2016/11/15/bunimovich-stadium/

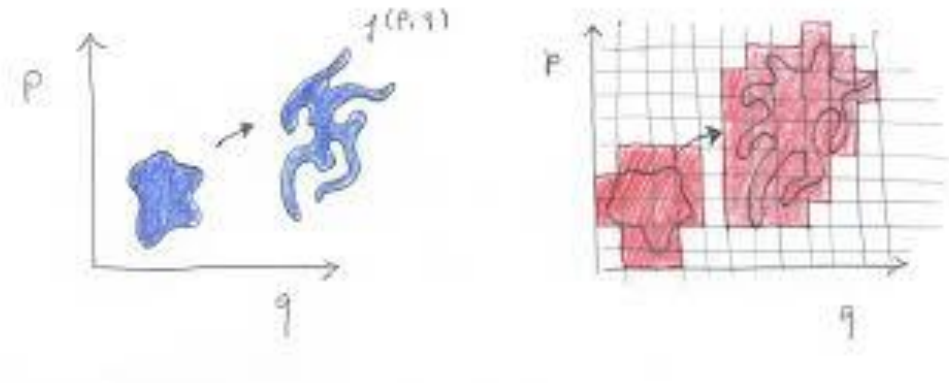


Problems with the Ergodic theorem

- In fact a path cannot cover a space so we can only hope that it comes close to a point.
- Liouville's theorem states that the volume remains constant, so how can it cover the whole phase space?

The solution is coarse graining

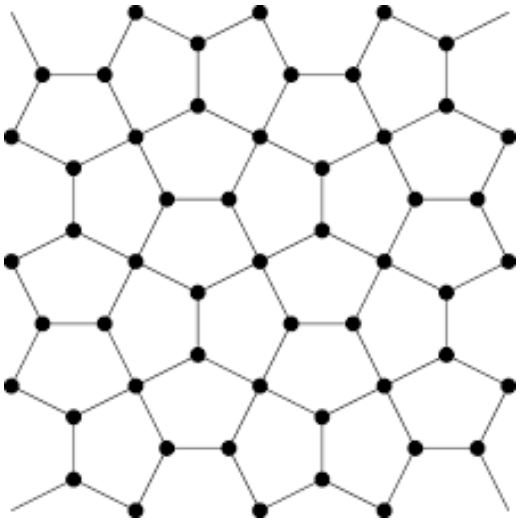




Coarse graining

We don't talk about points in the phase space rather neiborhoods at scales given by λ.

Once the flow passes through a cell we take that as counted.



$$\frac{ds}{dt} = \frac{d}{dt}\left(\mathrm{k}\ \mathrm{Log}(\Omega)\right) \geq 0$$

Ergodic Theorem

- Symmetry breaking means that the Phase space is no longer path connected, so Ergodic theorem must fail too

Critical exponents

- Changes of the physical observables near the critical point are given by power laws:

$$M \sim (-t)^{\beta}$$

$$t = \frac{T - T_c}{T_c}$$

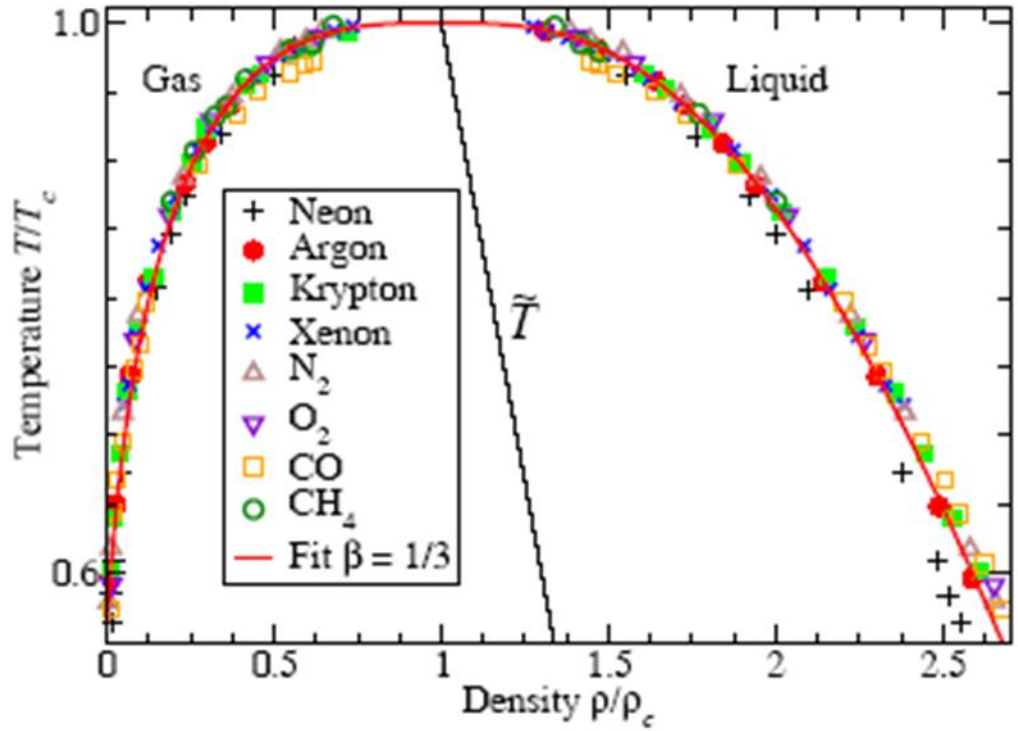
here M is magnetization, T temperature, T_c critical temperature, and t is called the reduced temperature

Critical exponents

Exponent	Physical quantity	expression
α	Specific heat (at zero field)	$C_v \sim t^{-\alpha}$
β	Order Parameter (below T_c ,zero field)	$m \sim (-t)^{\beta}$
γ	Susceptibility	$\chi \sim t^{\gamma}$
δ	Order Parameter at T_c (non zero field)	$m \sim h^{1/\delta}$
η	Correlation function at T_c	$C(r) \sim r^{2-d-\eta}$
ν	Correlation length	$\xi \sim t^{-\nu}$
z	Time scale τ	$\tau \sim \xi^{z}$

Universality

critical exponents of many seemingly very different physical systems, are the same



E. A. Guggenheim, J. Chem. Phys., vol. 13, p. 253, 1945.

Scale invariance

Many spin systems within statistical mechanics are characterized by a correlation length ξ, which is the scale by which two random variables remain in "contact" :

$$\mathrm{C(r)} = \langle \varphi(r)\varphi(0) \rangle \sim e^{-|r|/\xi}$$

However near the critical point we know that the correlation length diverges $\xi \sim t^{-\nu}$

Scale invariance

In this situation , distributions and correlations take on a power law format :

$$\langle \varphi(r)\varphi(0)\rangle \sim r^{2-d-\eta} \sim r^{-\eta}$$ (for d=2 T=Tc)

or

$$C_v(\lambda\, t) \sim \lambda^{-\alpha}\, C_v(\, t)$$

Scaling relations

$$F(h,t) = t^{2-\alpha}\, g(ht^{\Delta})$$

then:

$$C_v = \frac{\partial^2 f}{\partial t^2} \sim t^{-\alpha}$$

$$m = \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta}$$

$$\chi = \frac{\partial^2 f}{\partial h^2} \sim t^{-2+\alpha+\Delta}$$

Resulting in:

$$\alpha + 2\beta + \gamma = 2$$

Test of scaling relations

A. lense and M. Lagues, scale invariance: From Phase transitions to Turbulance. Berlin: Springer, 2008.

	Xe	Binary mixture
α	0	0.113
β	0.35	0.32
γ	1.3	1.239
δ	4.2	4.85
η	0.1	0.017
ν	0.57	0.625
$\alpha+2\beta+\gamma=2$	2.00	1.99
$\alpha+\beta\ \delta+\beta=2$	1.82	2.00
$\eta+\gamma/\nu=2$	2.38	2.00
$\alpha+\nu d=2$	1.71	1.99

The Renormalization Group (RG)

Dynamics at scale λ :

$$H = \sum_n C_n(\lambda)\, \widehat{O_n}$$

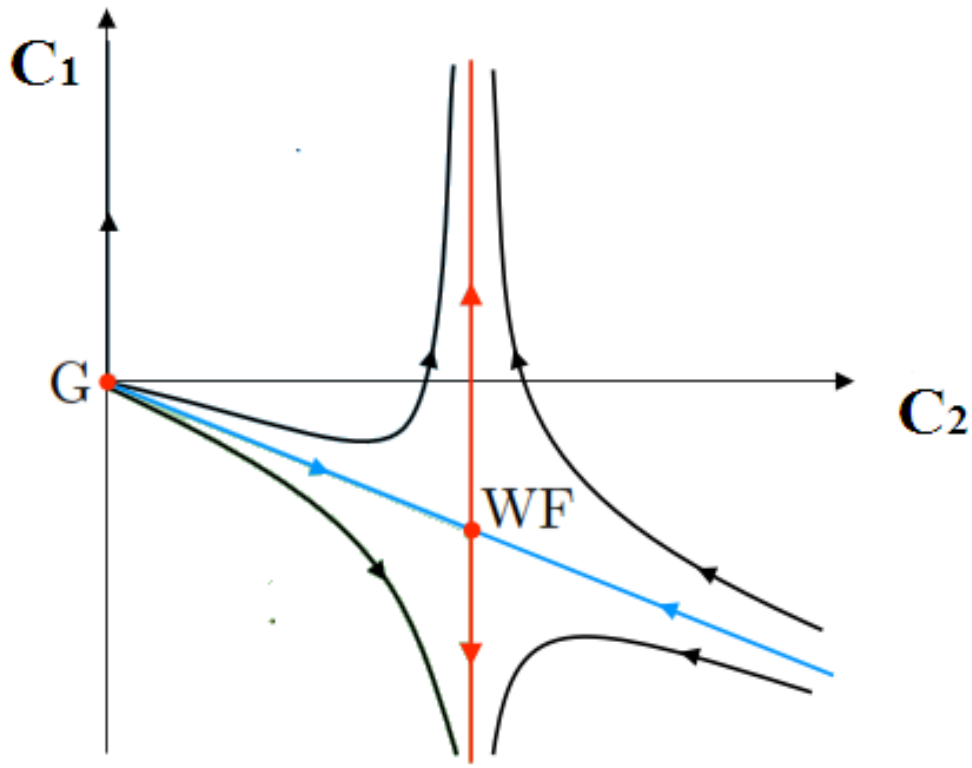
The RG Equation

$$\lambda \frac{dC_n}{d\lambda} = \beta_n(\vec{C}, \lambda)$$

The beta functions β_n will be given by the system in question.

It is possible that the beta functions admit a fixed point:

$$\beta_n(C^*, \lambda) = 0$$



A typical RG flow diagram. The Wilson-Fisher (WF) point is attractive in the IR limit in one direction only (blue line), the critical manifold. For the Gaussian point all directions are repulsive so no critical manifold exists. The black lines depict experimental observation.

Near the critical point

Expand RG equation near the fixed point:

$$\delta_n = C_n - C^*{}_n$$

$$\lambda \frac{d\delta_n}{d\lambda} = M_{n,m}\delta_m + \cdots$$

$$M_{n,m} = \partial_n \beta_m$$

The RG Equation

The eigenvalues of M are :

$$\lambda^{y_n}$$

Where $y_n > 0$ relevant operators

$y_n < 0$ irrelevant operators

$y_n > 0$ marginal operators

Scale invariance of the free energy

Now scale invariance of physical observables requires the free energy to have the form scalig form:

$$F(t,h) = t^{\frac{d}{y_t}} \mathrm{f}\left(\frac{h}{t^{y_h/y_t}}\right)$$

of course this applies to only systems which have just two relevant operators.

Scale invariance of the free energy

The two variables which are important are

- reduced temperature: t
- applied magnetic field: h

So expanding RG near the Gaussian point we must get :

$$t \rightarrow \lambda^{y_t} t$$
$$h \rightarrow \lambda^{y_h} h$$

Scale invariance of the free energy implies scaling relations

A simple calculation with the free energy on previous slide will determine the critical exponents. e.g.

$$C_v = \frac{\partial^2}{\partial T^2}\mathrm{F}$$

Hence at *h=0* we have :

$$C_v \sim t^{-\alpha}$$

$$\alpha = 2 - \frac{d}{y_t}$$

Scaling form of the free energy implies scaling relations

This observation has many consequences:

1-All critical exponents are determined by two quantities y_t and y_h .

Which means all critical exponents are not independent. In fact there are linear relations among them e.g.

$$\alpha + 2\beta + \gamma = 2$$

B. Widom, Equation of state in neighborhood of critical point. J. Chem. Phys. **43, 3898 (1965)**

Scaling form of the free energy implies scaling relations

Later Wilson showed that the scaling form of the free energy near Tc can be obtained from RG

The full set of scaling relations are:

$$\alpha + d\nu = 2$$
$$\alpha + 2\beta + \gamma = 2$$
$$\beta(\delta - 1) = \gamma$$
$$\nu(2 - \eta) = \gamma$$

K.G. Wilson, Phys. Rev. B 4, 3174 (1971); 4, 3184 (1971)

Scaling form of the free energy implies scaling relations

2. All physical systems that share the same RG fixed point will have the same set of critical exponents (Universality) . That is the details of their interactions are irrelevant (irrelevant operators)
3. For each universality class there exists infinitely many physical systems which share the same RG fixed point. (they differ in their irrelevant operators)

Next Lecture : fractals ,self-similarity , Conformal Field Theory