self-similarity, fractals, CFT

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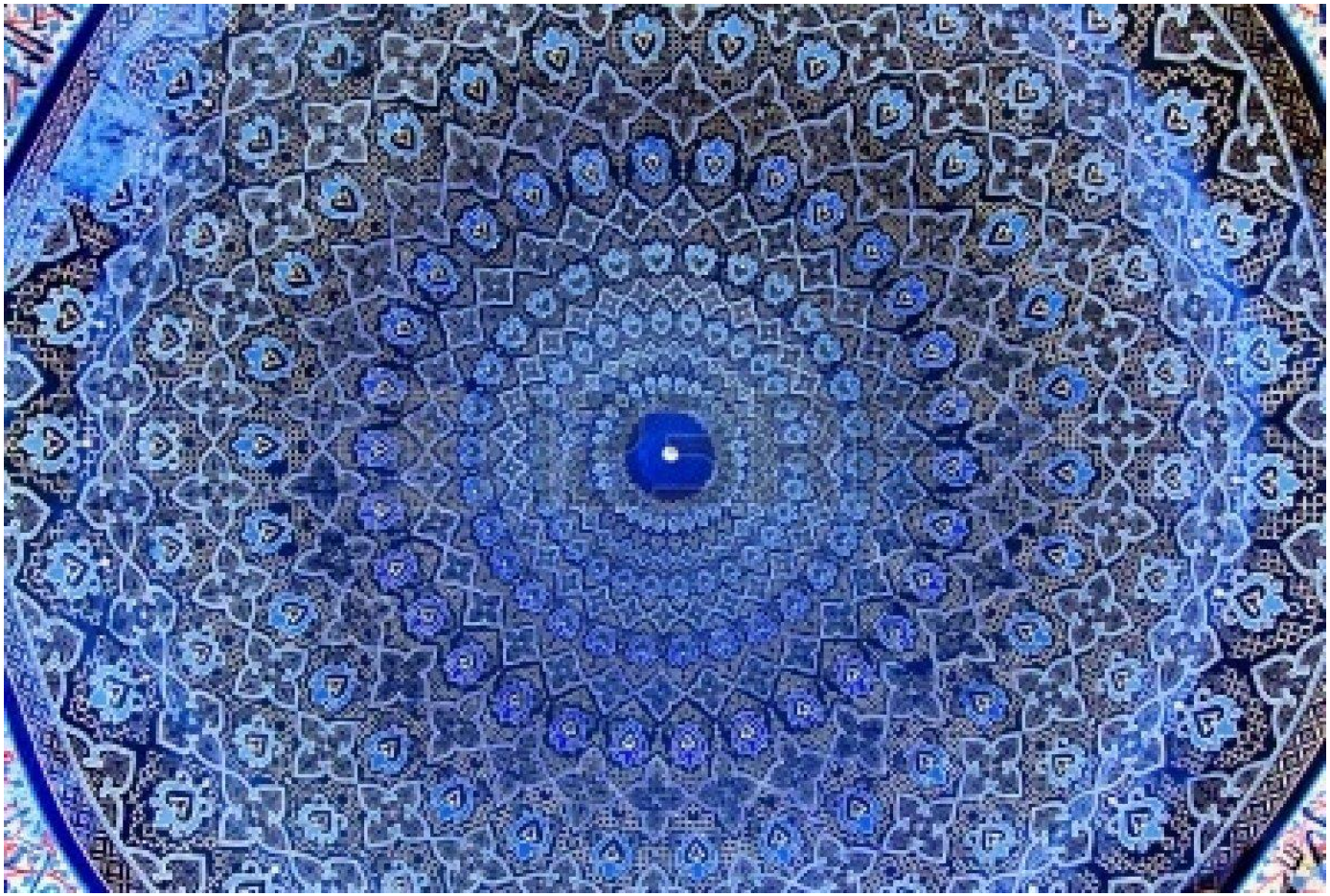
Overview

- Self-similarity
- Fractals
- Conformal mapping
- CFT
- Classification of critical phenomena

self similarity



Masjid-i-mam, Isfahan, Iran.

Self-Similarity

phase boundaries near the CP are scale invariant curves, may be fractals since they need to be self-similar

A geometrical shape is self-similar if it looks “almost" the same on any scale.

Fractals are self-similar objects.

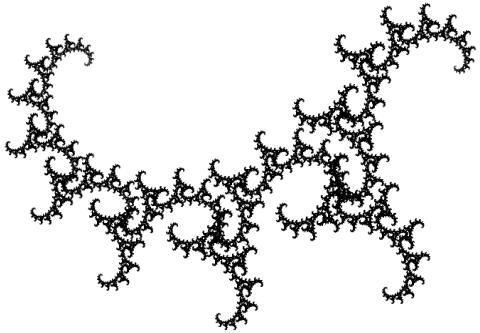
Self-affine vs. Self-similar

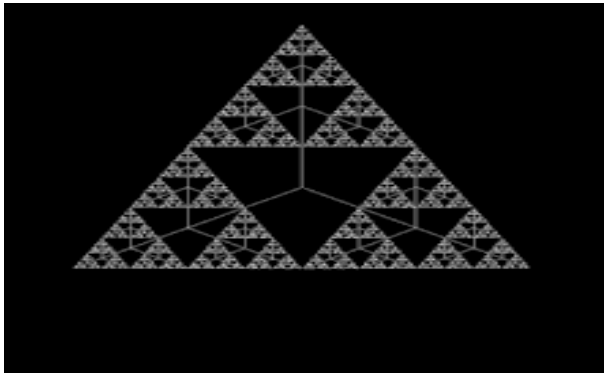
Self-affinity is a feature of a shape which when scaled by different amounts in different directions, the same shape is returned:

$$by(x) = y(a\,x)$$

Self-similarity happens when the scaling factors in both x and y directions are the same *a*=*b* .

Fractals





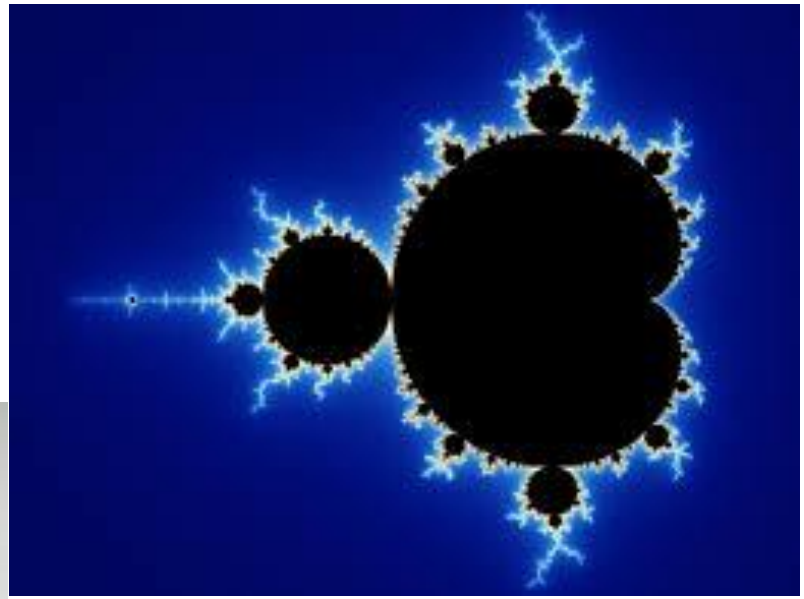
Fractals

curve or geometrical object, each part of which has the same statistical character as the whole.

Fixed point of repeated

Function application

F(F(….F(x)))



Mandelbrot Set

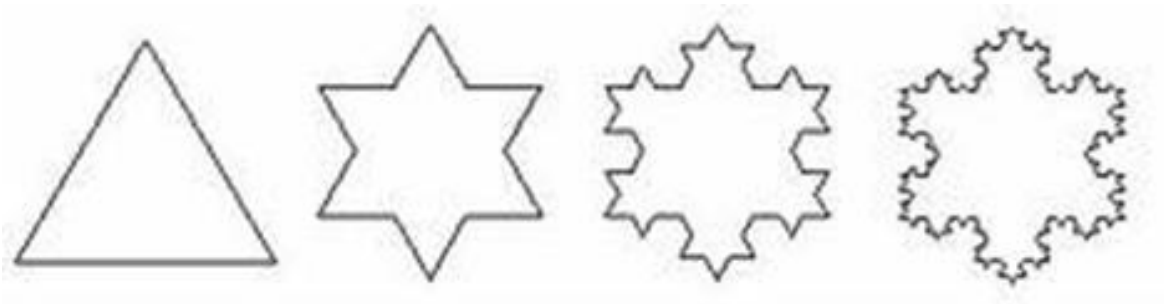
Fractal Dimension

- A fractal curve, has infinite lentgh if interpreted as a onedimensional object
- On the other hand it does not fill the embedding space
- A *fractal* is by definition a set for which the Hausdorff-Besicovitch *dimension* strictly exceeds the topological *dimension*.

Koch Snowflake

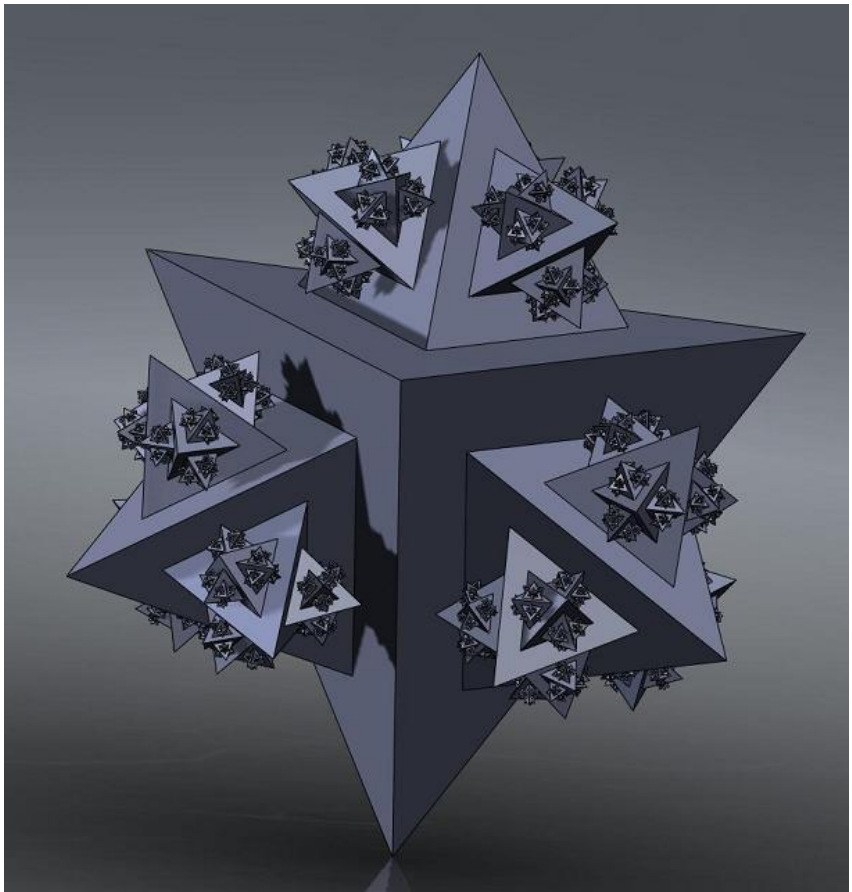
A famous example is the Koch Snowflake. The scale factors are 3 and 4

The fractal dimension of the Koch Snowflake is log(4)/log(3) ~ 1.26186



fractal

- Hence the Koch Snowflake is not a one dimensional object. It is a fractal with dimension d_f.



Length of the Koch Snowflake

Number of sides at each iteration

$$N_k = 4N_{k-1} = 3.4^k$$

Length of a side at each iteration

$$s_k = l/3^k$$

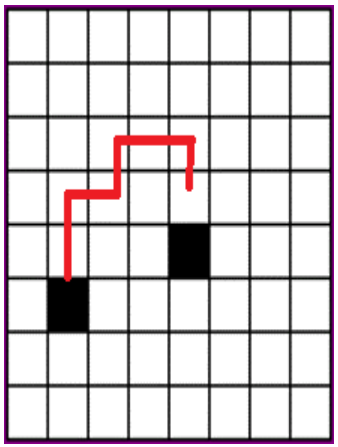
Total length: $\lim_{k\to\infty} \frac{N_k l}{3^k} = 3l * (\frac{4}{3})^k \to \infty$

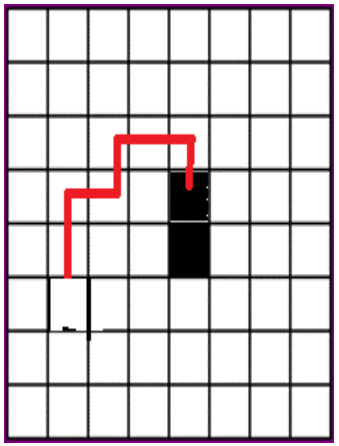
Calculate the fractal dimension using correlation methods

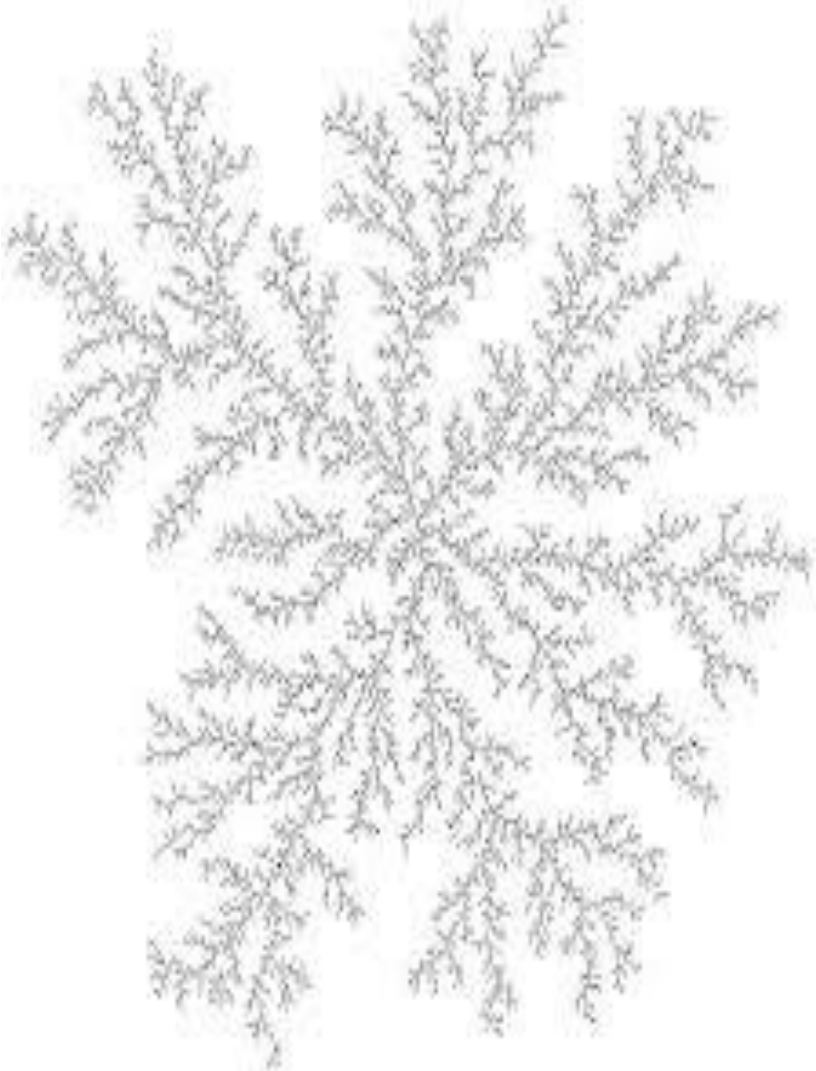
d_f=1.539..

Diffusion Limited Aggregate

Aggregation of particles by random walk







Calculate the fractal dimension using correlation methods

Box counting method ,

number of boxes in radius R is

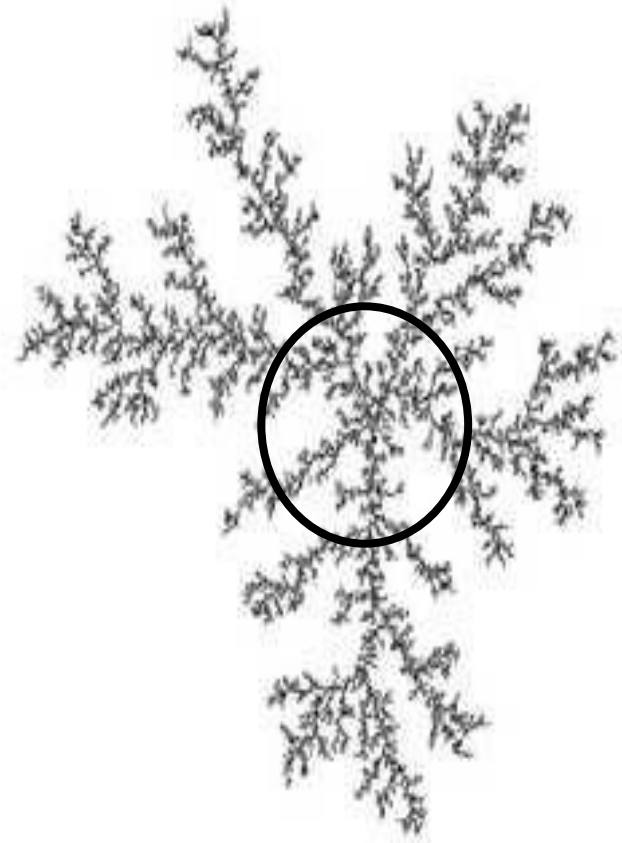
$N(R) \sim R^{d_f}$

number of boxes of size *a* is :

$N(R) \sim a^{-d_f}$

Or summing yup

$N(a, R) \sim \ (R/a)^{d_f}$



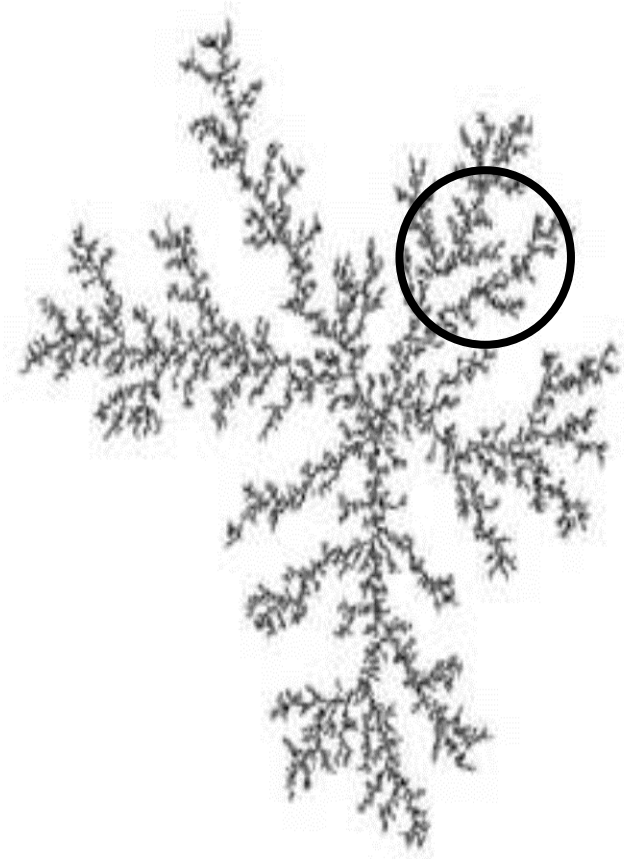
Calculate the fractal dimension using correlation methods

What if the center changes ?

$$C(r,x) = \langle \rho(x+r)\rho(x) \rangle$$

average density of particles arc
the particle at x

$$C(r) = \int C(r,x)\, dx \sim r^{d_f -}$$



Calculate the fractal dimension of images

Density autocorrelation method allows us to calculate the fractal dimension of any image !

Estimate fractal dimension of “entities” in 3d

using gray scales on 2d images of them.

d_f=1.334

d_f=2.867

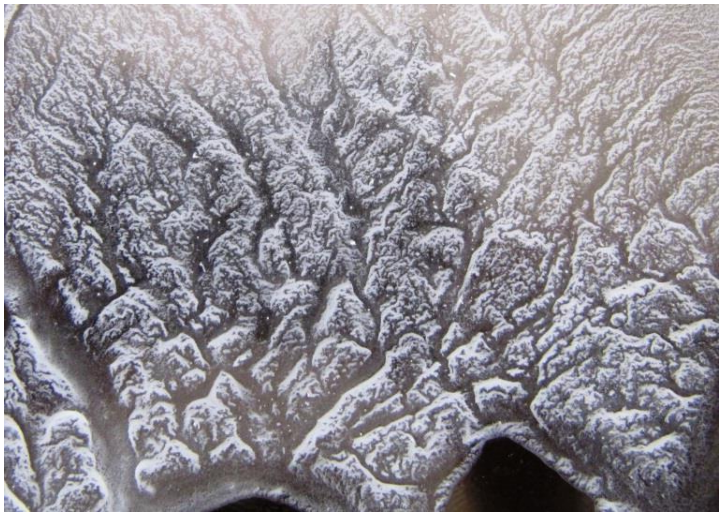
- M Ghafari, M Ranjbar, S Rouhani; “Observation of a crossover in kinetic aggregation of Palladium colloids”
  Applied Surface Science, 2015 , 1143-1149 ; arXiv preprint arXiv:1412.8052 353,
- Shanmugavadivu, P., and V. Sivakumar. "Fractal dimension based texture analysis of digital images." Procedia Engineering 38 (2012): 2981-2986.

Hausdorff dimension

What is also referred to as the “Fractal Dimension” of a curve, is usually the Hausdorff dimension, or the scale by which the whole curve scales as we reduce the size of the measuring stick :

$$N \sim s^{d_f}$$

Fractals

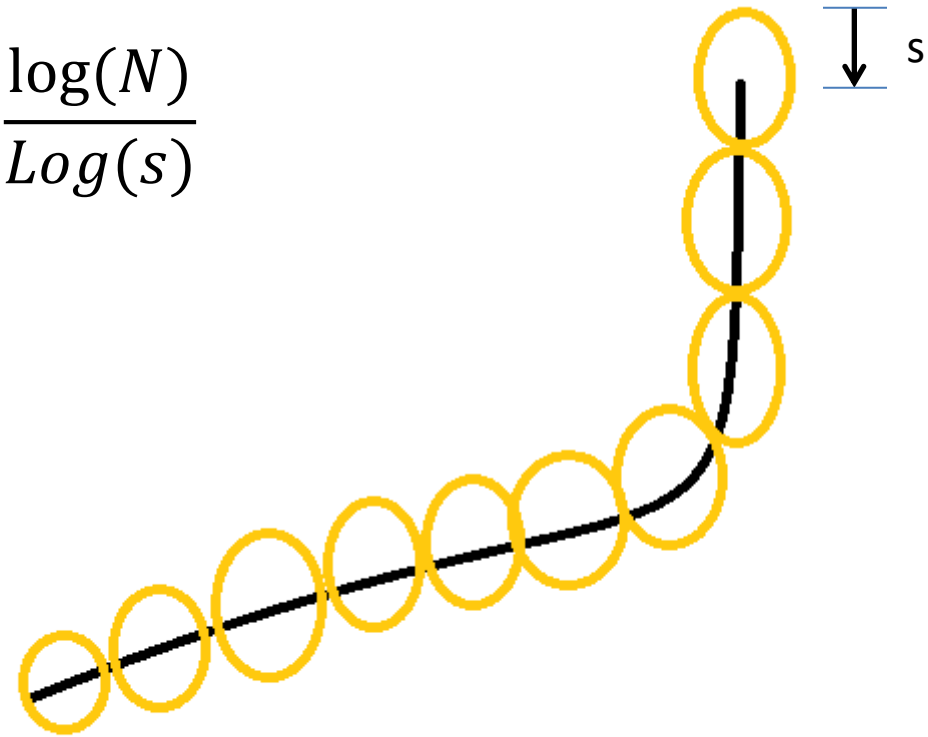


$$d_f \approx 1.22$$

Fractal flow of inhomogeneous fluids over smooth inclined surfaces and determination of their fractal dimensions and universality classes
N Maleki-Jirsaraei, B Ghane-Motlagh, S Baradaran, E Shekarian, S Rouhani
Journal of Physics: Condensed Matter 17, S1209

Calculation of Hausdorff dimension of a shape

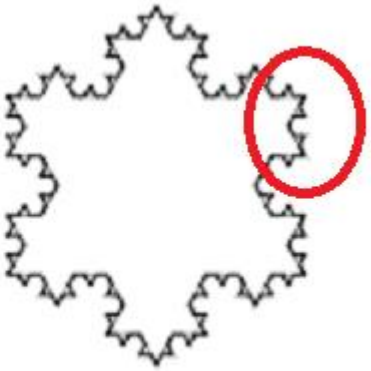
$$d_f = \lim_{s \to 0} -\frac{\log(N)}{Log(s)}$$



Fractal dimension:

$$d_f = \frac{\log(3)}{\log(2)} = 1.58..$$

$$d_f = \frac{\log(4)}{\log(3)} = 1.26..$$



Hurst exponent

$$x(\lambda t) = \lambda^H x(t)$$

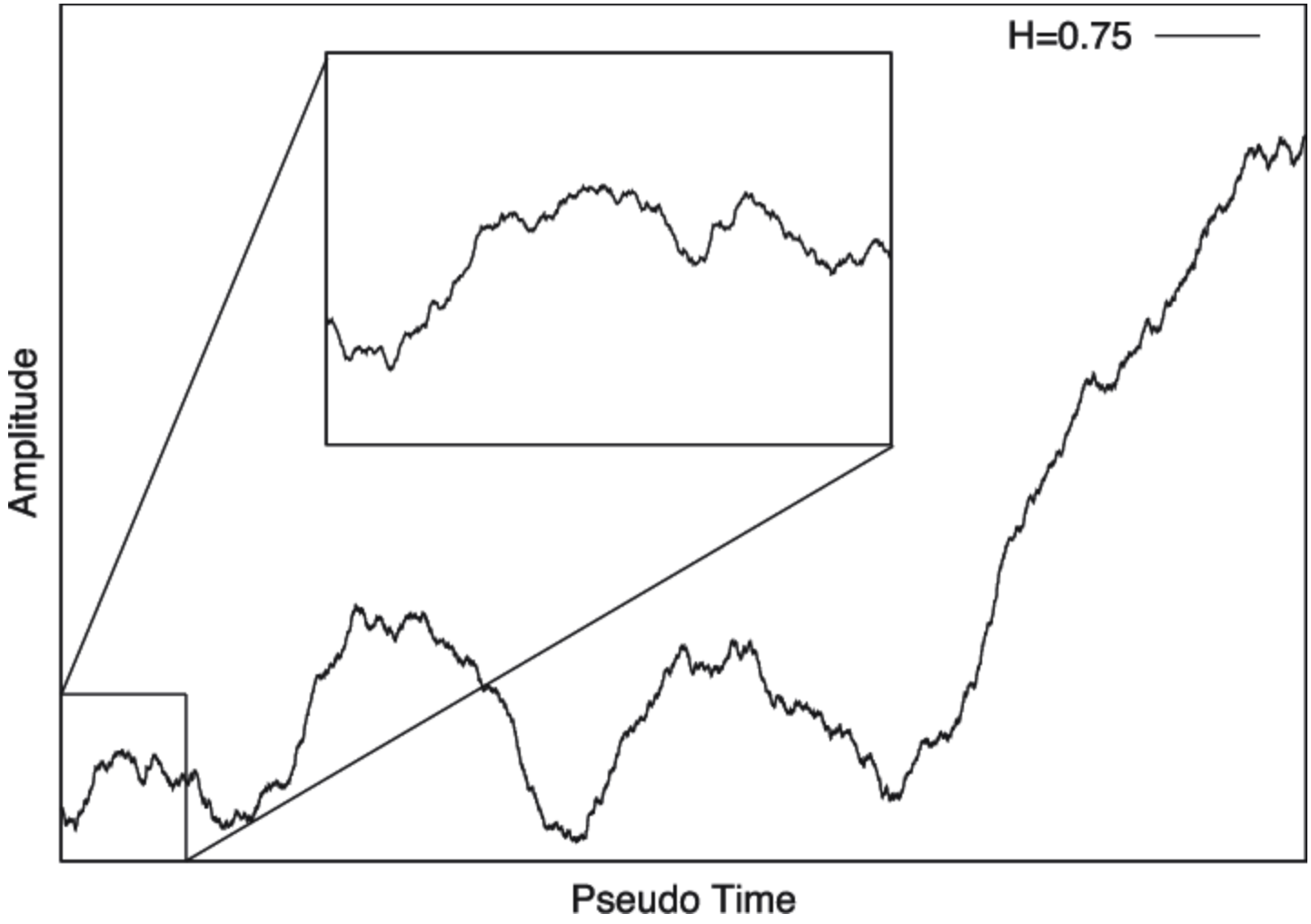
A measure of self similarity in time series.

$$H = 2 - d_f$$

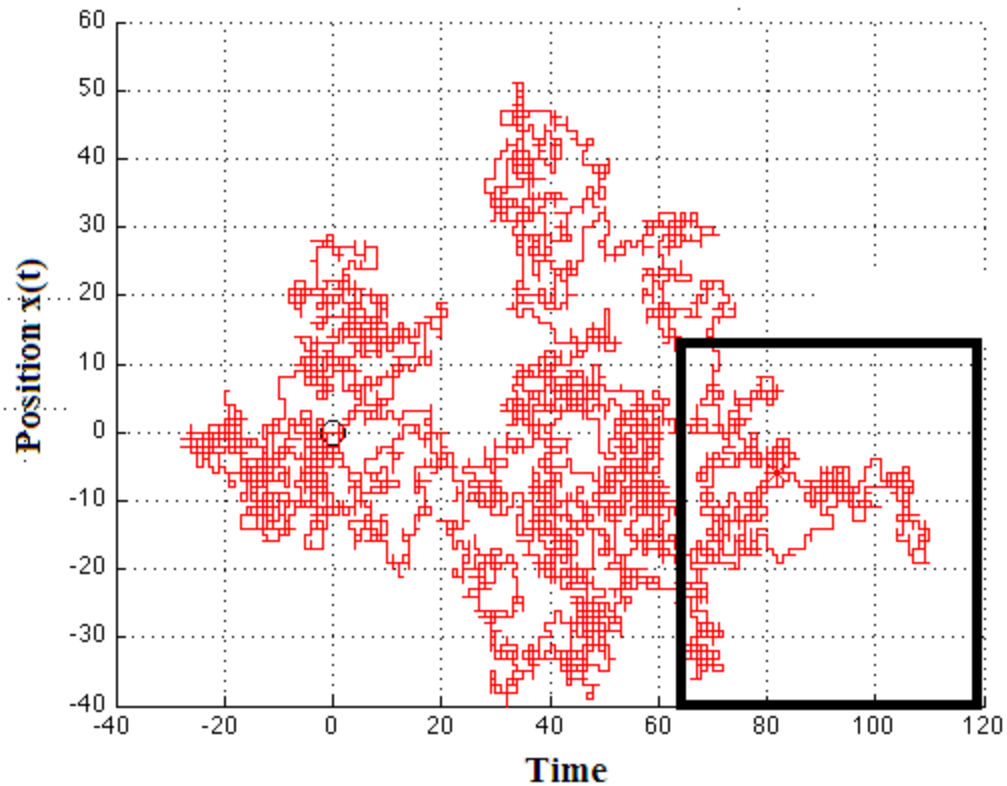
Time series classification;

$H > 1/2$	**Long range dependence**
$H = 1/2$	Brownian motion
$H < 1/2$	negatively correlated

G. A. Maclachlan et al, The Hurst exponent of Fermi gamma-ray bursts, Monthly Notices of the Royal Astronomical Society 436 (2013)2907-2914



Random fractals: roughly the same

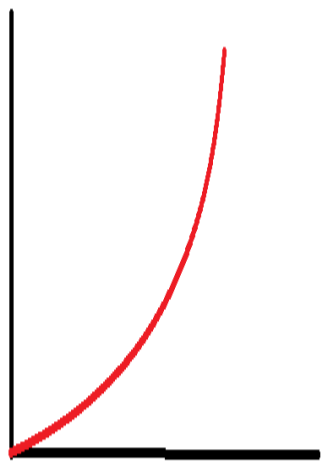


Random fractal curve

Scaling Graphs

$y(ax) = a\, y(x)$ self-similar

$y(ax) = b\, y(x), \quad b \neq a$ self-affine



Not Interesting !

$eg\ Y(x) = c\, x^3\,, \qquad b = a^{-3}$

Weierstrass function

$$y(x) = \sum_{n=0}^{n=\infty} b^n \,\mathrm{Cos}(a^n x)$$

b is positive odd integer such that

$$ab > 1 + \frac{3}{2}\pi$$

- Continuous everywhere
- nowhere differentiable

Graph of Weierstrass function

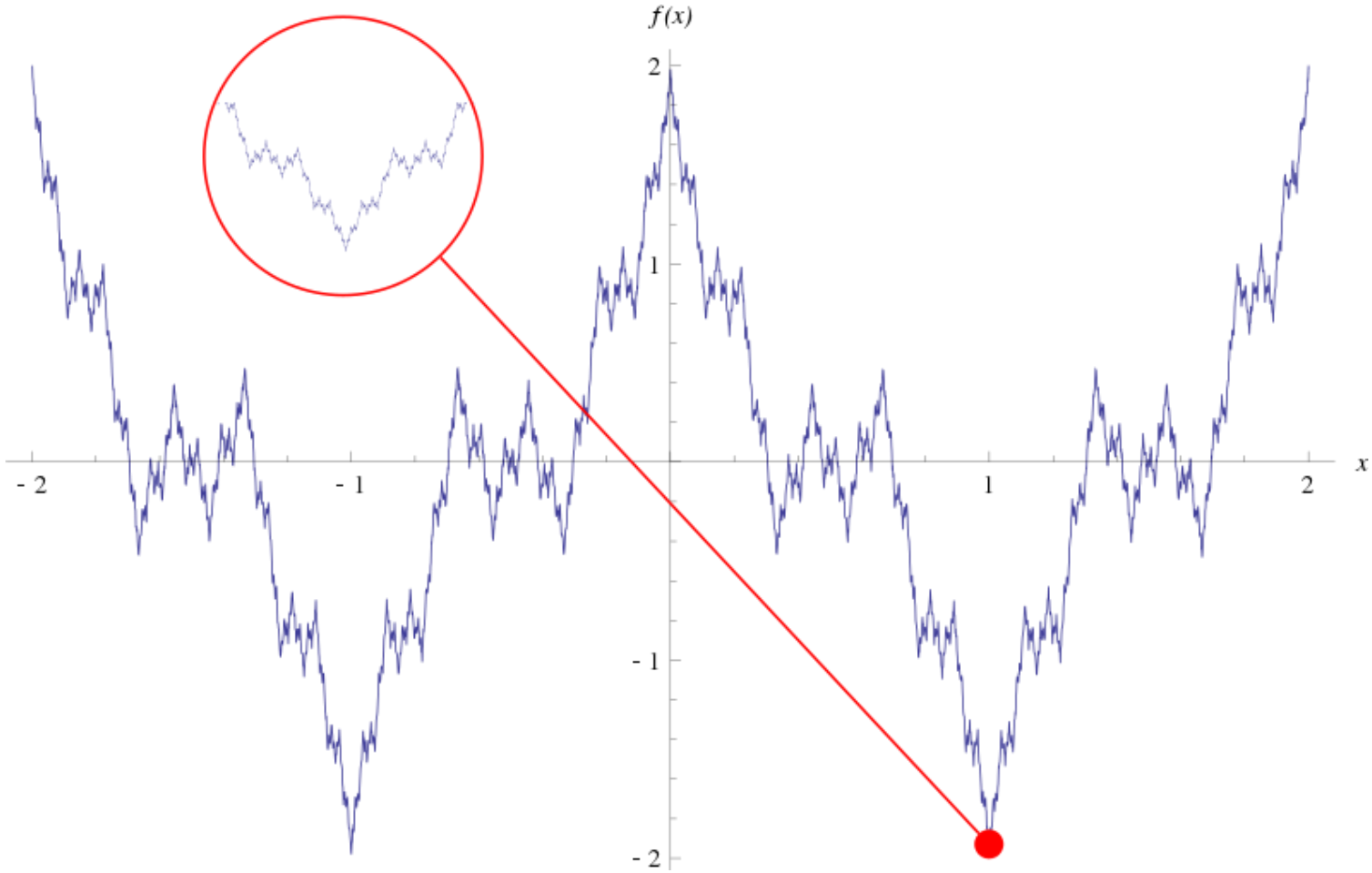


Image wikipedia

Weierstrass function is self-affine

$$b\, y(a\, x) = y(x)$$

Hausdorff dimension:

$$D = 2 + \log(b)\, / \log(a)$$

Clearly *1<D<2*

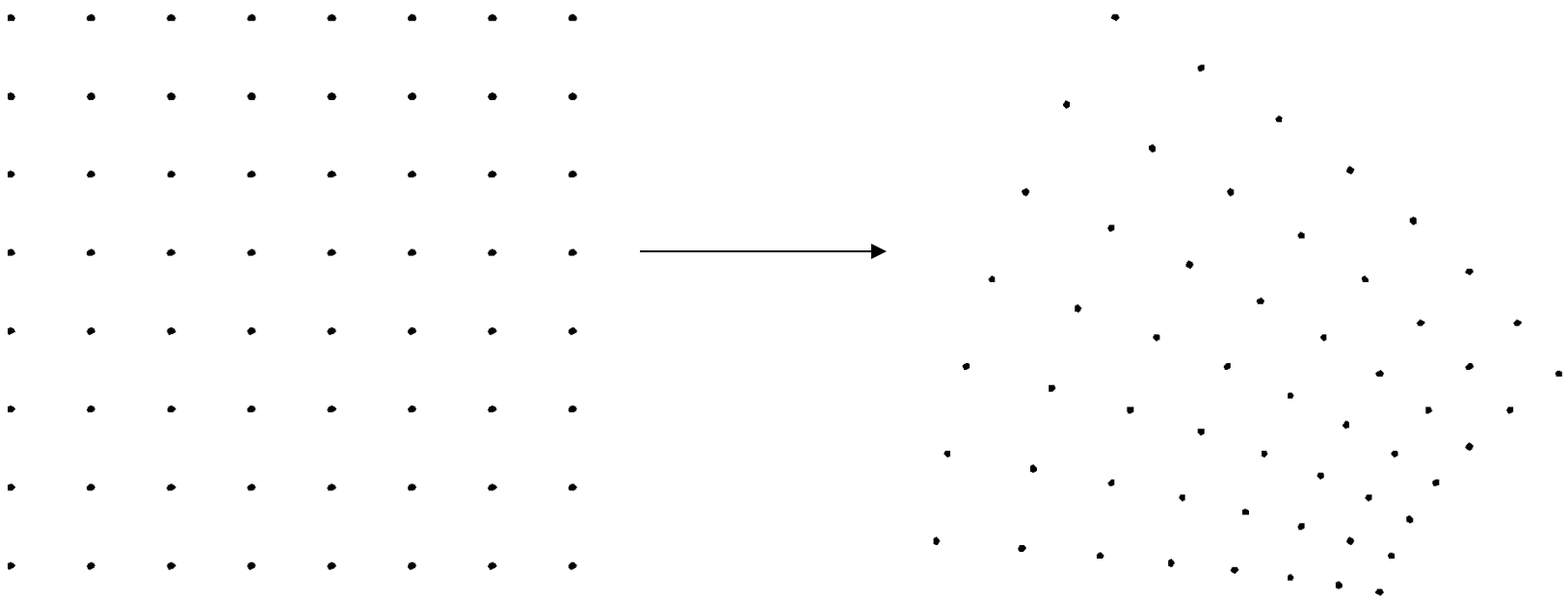
Hunt, Brian. "The Hausdorff dimension of graphs of Weierstrass functions." *Proceedings of the American mathematical society* 126.3 (1998): 791-800.

Conformal mapping, holomorphicity and Riemann's theorem

A conformal transformation is one which preserves angles locally. In other words changes the metric up to an overall factor:

$$g_{\mu\nu} \rightarrow \rho\, g_{\mu\nu}$$

$$g_{\mu\nu} \rightarrow \rho\, g_{\mu\nu}$$



Translations+Rotations+Scale+Inversion
angles remain unchanged

Complex Coordinates

Existence of scale invariance at the CP may lead to conformal invariance, with far reaching consequences for critical phenomena.

In 2d things are much easier. Let us begin by complexifying the coordinates:

$$z = x + iy, g: z \to g(z)$$

therefore

$$\bar{\partial}\, g(z) = 0$$

Conformal mappings of the complex plane

Cauchy-Riemann Conditions

$$g(z) = u(x, y) + iv(x, y)$$

then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

or

$i\,\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y}$ can be rewritten as $\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) g = 0$

Alternatively written as $\bar{\partial} g = 0$, since $\bar{\partial} z = 0$ in other words f is not a function of $\bar{z}$, hence a holomorphic function.

Conformal mappings of a domain D in the complex plane

$$z \to f(z)$$

- A holomorphic function is a conformal mapping if $f' \neq 0 \; on \; whole \; of \; D$.

- This means that it is one-one and holomorphic.

Conformal transformation

conformal group

Conformal transformations form a group named the **conformal group**

which is an extension of the group of iso-metries of space-time.

For the case of Poincare invariance in d dimensions, the conformal group is SO(d+2) d>2

When d=2 we have the Mobius group.

Mobius group

Mobius transformations:

$$z \rightarrow \frac{az+b}{cz+d} \qquad , ad\text{-}bc=1$$

Are one-one and onto so reversible transformations of the complex plane

Inverse map is:

$$\frac{-dz+b}{cz-a}$$

These transformations form the group SL(2,c).

Mobius group

The Mobius transformation includes all the well-known isometries of the plane:

translations (*a=1, c=0, d=1*), $z \to z + b$

rotations and scaling (*b=0, c=0, d=1/a*) $z \to a^2 z$

inversion (*a=0, d=0, b=-1/c*) $z \to -\frac{1}{c^2 z}$

2d Conformal Group

In physics (d=2) all holomorphic transformations of the complex plane become admissible.

(no need to be bijective)

so one ends up with an infinite dimensional symmetry, leading to the Witt and Virasoro Algebras.

Mapping the complex plane onto itself

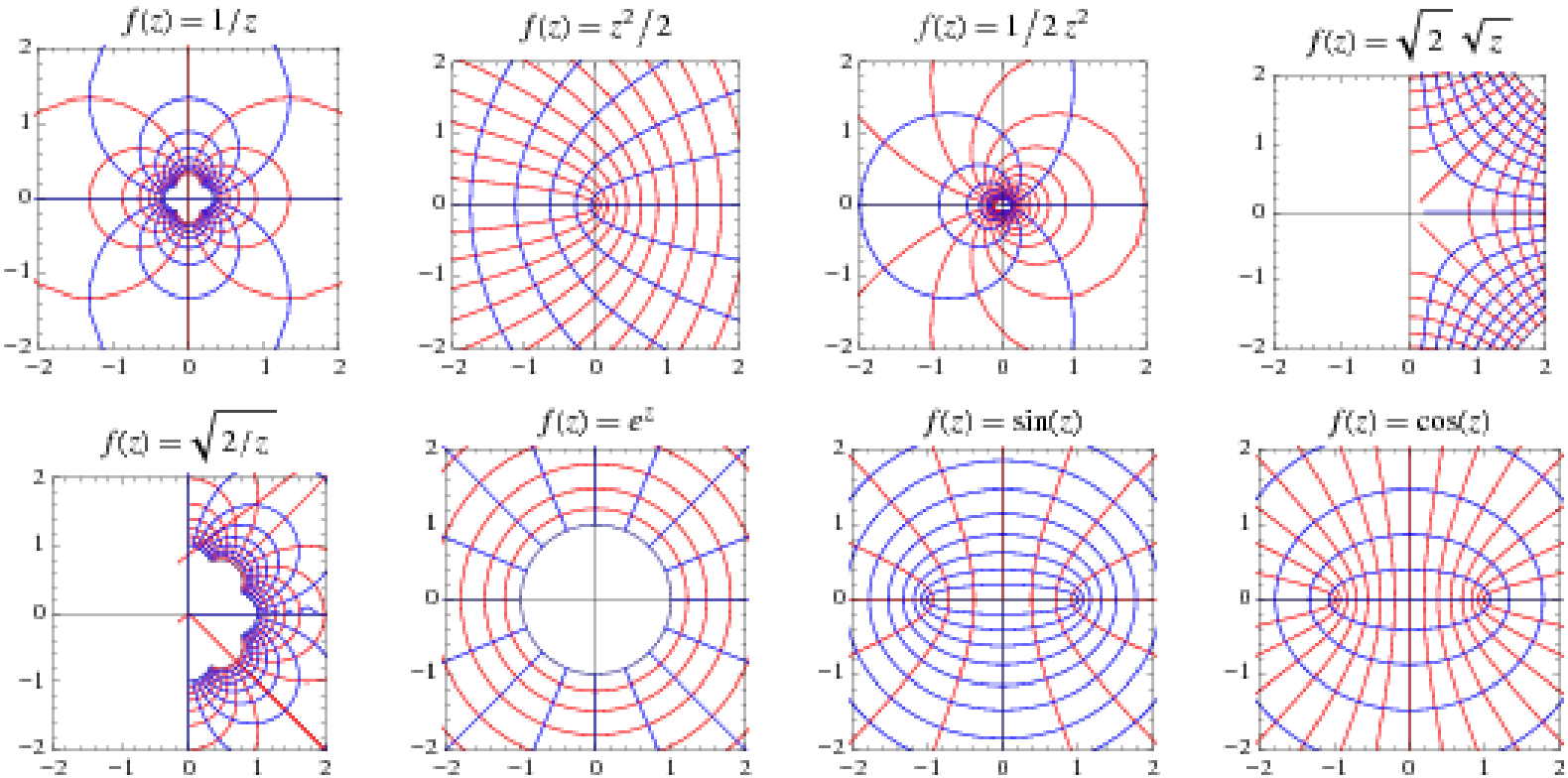
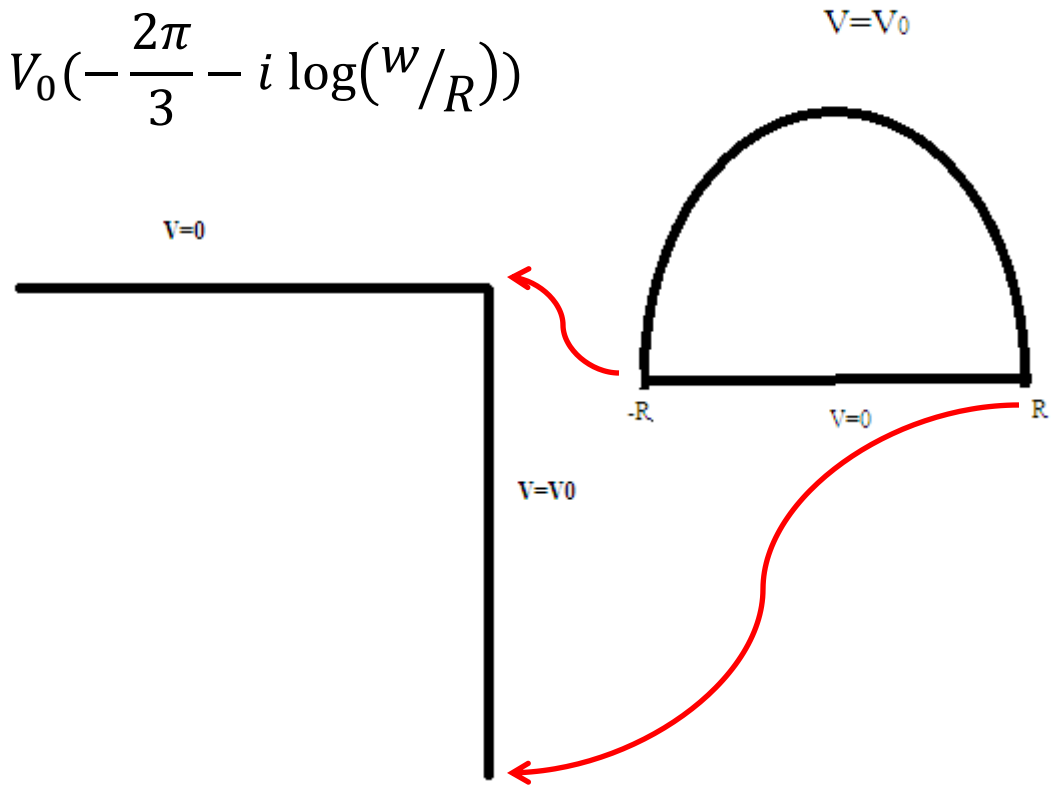


Image credit: Wolfram Mathematica

Solution of some electrostatic problem

The electric potential v(x,y) on the inside of a semi-disc of radius R; such that on the upper part of the disc it has value $v=v_0$, and value v=0 on the interval [-R, R].

$$w(z) = \frac{z+R}{z-R} \qquad V(w) = V_0\left(-\frac{2\pi}{3} - i\log\left({}^{w}/_{R}\right)\right)$$



Take the branch of log such that we are in the $(-\pi, +\pi)$ sector.

Sl(2,c) Algebra

Mobius transformations:

$$z \to \frac{az+b}{cz+d} = \omega(z),\ ad\text{-}bc=1$$

Are one-one and onto so reversible.

Inverse map is:

$$\frac{-d\omega + b}{c\omega - a} = z$$

These transformations form the group SL(2,c).

Witt Algebra

Note that sl(2,c) algebra $L_{0,}\, L_{\pm}$ sits in the center of Witt algebra; (look at the Laurant expansion)

$$L_n = -z^{n+1}\frac{\partial}{\partial z}$$

$$[L_{0,}\, L_{\pm}] = \mp L_0$$
$$[L_+, L_-\,] = 2L_0$$
$$[L_n, L_m] = (n-m)\, L_{n+m}$$

Why Conformal Symmetry

- We know that critical point (CP) has scale invariance
- We also know that scale invariance + Physical theory leads to conformal invariance (2d)
- Thus if the "scaling limit" for a lattice model holds to each critical point we should be able to assign a Conformal Field Theory
- Conformal symmetry is translations + rotations+scaling+inversion

Scale *vs* conformal invariance

Scale invariance plus some physical properties implies conformal invariance of the physical system:

a) Conservation of energy-momentum
b) Unitarity
c) Locality
d) Discrete spectrum in scaling dimensions

Has a proof in 2d but may also be true in 4d

Yu Nakayama. Scale invariance vs conformal invariance. Physics Reports. 2015, Vol. 569, pp. 1-93.

Scale vs. Conformal invariance

In 2d the energy-momentum tensor has three components. In complex coordinates these are:

$$\mathrm{T} = \mathrm{T}_{\mathrm{zz}}, \overline{\mathrm{T}} = \mathrm{T}_{\overline{\mathrm{zz}}}, \Theta = \mathrm{T}^{\mu}_{\mu}$$

conservation of energy momentum tensor is:

$$\bar{\partial}\mathrm{T} + 4\partial\Theta = 0$$

Scale vs. Conformal invariance

Scale invariance means that the trace of energy momentum tensor vanishes:

$$\Theta = \mathrm{T}^{\mu}_{\mu} = 0$$

conservation of energy momentum tensor is:

$$\bar{\partial}\mathrm{T} = 0$$

Thus *T=T(z)* only.

CFT_2 is fully integrable

- Hence

$$\bar{\partial}\mathrm{T} = 0 \rightarrow T(z)$$

In other words we have invariance under holomorphic transformations $z \rightarrow f(z)$.

Also the theory is fully integrable:

$$\bar{\partial}\mathrm{T}^n = 0 \qquad \forall\, n \epsilon \mathbb{N}$$

Scale vs. Conformal invariance

- Therefore we are able to correspond to every CP (scale invariant) a CFT if the conditions above hold.

- As we shall see these conditions although sufficient but they are not necessary as there are examples falling outside of them.

The scaling limit

In the limit that the lattice spacing tends to zero compared to relevant length scales of the theory (e.g. correlation length) we will have a field theory corresponding to the statistical physics system.

Example: the Ising model in the scaling limit corresponds to the free fermion theory which become critical if m vanishes (m~ t):

$$\frac{i}{2}\int\{\psi_1\bar{\partial}\,\psi_1+\psi_2\partial\psi_2+m\,\psi_1\psi_2\}$$

Conformal Field Theory

What this result implies is that in the scaling limit any critical, unitary … statistical physics model will correspond to a conformally invariant field theory (CFT)

Field operators

Under conformal transformations $z \to w(z)$, all field operators must be representations of the Virasoro algebra.

Under a conformal transformation field operators transform as

$$\varphi(w, \overline{w}) \ \to (\frac{\partial w}{\partial z})^{-h} \, (\frac{\partial \overline{w}}{\partial \overline{z}})^{-\overline{h}} \, \varphi(z, \overline{z})$$

Conformal Field Theory - CFT

These are called quasi-primary fields. The conformal weights h and $\overline{h}$ are related to the scaling dimension Δ and spin s:

$$h = \frac{1}{2}(\Delta + s) \qquad \overline{h} = \frac{1}{2}(\Delta - s)$$

2,3 point functions

$$\langle \varphi_{h_1}(z_1)\varphi_{h_2}(z_2)\rangle = \frac{\delta_{h_1,h_2}}{|z_1 - z_2|^{h_1+h_2}}$$

$$\langle \varphi_{h_1}(z_1)\varphi_{h_2}(z_2)\varphi_{h_3}(z_3)\rangle = \frac{C_{123}}{{x_{12}}^a {x_{23}}^b {x_{31}}^c}$$

$$a = h_1 + h_2 - h_3, \ldots$$

$$x_{12} = z_1 - z_{2,\ldots}$$

4 point function

Four point function is

$$\langle \varphi_{h_1}(z_1)\varphi_{h_2}(z_2)\varphi_{h_3}(z_3)\varphi_{h_4}(z_4)\rangle$$
$$= f(\eta)\prod_{i<j} x_{ij}^{\frac{h}{3}-h_i+h_j}$$

$$h = h_1 + h_2 + h_3 + h_4$$

The only independent cross ratio is;

$$\eta = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

4 point function

- Symmetry does not determine the function $f(\eta)$
- To do so we need to specify the exact CFT we are dealing with then f satisfies a hyper-geometric function which is actually the expression of a <u>null state</u>

Minimal series

$$c = 1 - \frac{6}{m(m+1)}\ , m = 2,3,..$$

$\varphi_{p,q}$ has conformal dimensions:

$$h_{p,q} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)},\ 1 \leq p \leq m-1\ , 1 \leq q \leq p$$

Minimal Series

Other low lying CFT's in the minimal series are:

Table 1: Low lying CFT's and corresponding critical models in 2d.

m	c	Statistical model
3	1/2	Ising model
4	7/10	Tricritical Ising model
5	4/5	3-state Potts model
6	6/7	Tricritical 3-state Potts model

Question

Can you write a neural net that

1. Decides if an image is a fractal
2. Decides if it is a 3-D fractal or 2-D fractal
3. Calculates the fractal dimension.

https://aboveintelligent.com/what-do-deep-neural-networks-understand-of-fractals-2ae354911601